HYPERBOLICITY OF THE GRAPH OF NON-SEPARATING MULTICURVES

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Abstract. A non-separating multicurve of a surface S of finite type is a multicurve c so that $S-c$ is connected. For $k\geq 1$ define the graph $\mathcal{NC}(S,k)$ of non-separating k-multicurves to be the graph whose vertices are non-separating multicurves with k components and where two such multicurves are connected by an edge of length one if they can be realized disjointly and differ by a single component. We show that if k is smaller than the genus of S then $\mathcal{NC}(S,k)$ is hyperbolic.

1. Introduction

The *curve graph* $\mathcal{CG}$ of an oriented surface S of genus $g\geq 0$ with $m\geq 0$ punctures and $3g-3+m\geq 2$ is the graph whose vertices are isotopy classes of essential (i.e. non-contractible and not homotopic into a puncture) simple closed curves on S. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite δ-hyperbolic geodesic metric space of infinite diameter [MM99] for a number $\delta>0$ not depending on the surface [HPW13].

The *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving homeomorphisms of S acts on $\mathcal{CG}$ as a group of simplicial isometries. This action is *coarsely transitive*, i.e. the quotient of $\mathcal{CG}$ under this action is a finite graph. Curve graphs and their geometric properties turned out to be an important tool for the investigation of the geometry of $\mathrm{Mod}(S)$ [MM00].

If the genus g of S is positive then for each $k\leq g$ we can define another $\mathrm{Mod}(S)$-graph $\mathcal{NC}(S,k)$ as follows. Vertices of $\mathcal{NC}(S,k)$ are *non-separating k-multicurves*, i.e. multicurves ν consisting of k components such that $S-\nu$ is connected. Two such multicurves are connected by an edge of length one if they can be realized disjointly and differ by a single component. The mapping class group of S acts coarsely transitively as a group of simplicial isometries on the graph of non-separating k-multicurves. In fact, the action is transitive on vertices. Note that $\mathcal{NC}(S,1)$ is just the complete subgraph of $\mathcal{CG}$ whose vertex set consists of all non-separating simple closed curves in S.

The goal of this note is to show

Theorem. *For $k<g$ the graph $\mathcal{NC}(S,k)$ of non-separating k-multicurves is hyperbolic.*

For the proof of the theorem, we adopt a strategy from [H13]. Namely, we begin with showing that for $g\geq 2$ the graph $\mathcal{NC}(S,1)$ is hyperbolic. This is easy if S

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has at most one puncture, in fact in this case the inclusion map is a quasi-isometry (see Section 3). In the case that S has at least two punctures we use a tool from [H13]. This tool is also used in Section 4 to successively add components to the multicurve until the number $k < g$ of components is reached. We summarize the results from [H13] which we need in Section 2.

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2. Hyperbolic extensions of hyperbolic graphs

In this section we consider any (not necessarily locally finite) metric graph $(\mathcal{G}, d)$ (i.e. edges have length one). For a given family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs of $\mathcal{G}$ define the *$\mathcal{H}$-electrification* of $\mathcal{G}$ to be the metric graph $(\mathcal{E}\mathcal{G}, d_{\mathcal{E}})$ which is obtained from $\mathcal{G}$ by adding vertices and edges as follows.

For each $c \in \mathcal{C}$ there is a unique vertex $v_c \in \mathcal{E}\mathcal{G} - \mathcal{G}$. This vertex is connected with each of the vertices of H_c by a single edge of length one, and it is not connected with any other vertex.

Definition 2.1. For a number $r > 0$ the family $\mathcal{H}$ is called *r-bounded* if for $c \neq d \in \mathcal{C}$ the intersection $H_c \cap H_d$ has diameter at most r where the diameter is taken with respect to the intrinsic path metric on H_c and H_d.

A family which is r-bounded for some $r > 0$ is simply called *bounded*.

In the sequel all parametrized paths γ in $\mathcal{G}$ or $\mathcal{E}\mathcal{G}$ are supposed to be *simplicial*. This means that the image of every integer is a vertex, and the image of an integral interval $[k, k+1]$ is an edge or a single vertex.

Call a simplicial path γ in $\mathcal{E}\mathcal{G}$ *efficient* if for every $c \in \mathcal{C}$ we have $\gamma(k) = v_c$ for at most one k. Note that if γ is an efficient simplicial path in $\mathcal{E}\mathcal{G}$ which passes through $\gamma(k) = v_c$ for some $c \in \mathcal{C}$ then $\gamma(k-1) \in H_c, \gamma(k+1) \in H_c$.

Definition 2.2. The family $\mathcal{H}$ has the *bounded penetration property* if it is r-bounded for some $r > 0$ and if for every $L > 0$ there is a number $p(L) > 2r$ with the following property. Let γ be an efficient L-quasi-geodesic in $\mathcal{E}\mathcal{G}$, let $c \in \mathcal{C}$ and let $k \in \mathbb{Z}$ be such that $\gamma(k) = v_c$. If the distance in H_c between $\gamma(k-1)$ and $\gamma(k+1)$ is at least $p(L)$ then every efficient L-quasi-geodesic γ' in $\mathcal{E}\mathcal{G}$ with the same endpoints as γ passes through v_c. Moreover, if $k' \in \mathbb{Z}$ is such that $\gamma'(k') = v_c$ them the distance in H_c between $\gamma(k-1), \gamma'(k'-1)$ and between $\gamma(k+1), \gamma'(k'+1)$ is at most $p(L)$.

Let $\mathcal{H}$ be as in Definition 2.2. Define an *enlargement* $\hat{\gamma}$ of an efficient simplicial L-quasi-geodesic $\gamma : [0, n] \to \mathcal{E}\mathcal{G}$ with endpoints $\gamma(0), \gamma(n) \in \mathcal{G}$ as follows. Let $0 < k_1 < \cdots < k_s < n$ be those points such that $\gamma(k_i) = v_{c_i}$ for some $c_i \in \mathcal{C}$. Then $\gamma(k_i - 1), \gamma(k_i + 1) \in H_{c_i}$. For each $i \leq s$ replace $\gamma[k_i - 1, k_i + 1]$ by a simplicial geodesic in H_{c_i} with the same endpoints.

For a number $k > 0$ define a subset Z of the metric graph $\mathcal{G}$ to be *k-quasi-convex* if any geodesic with both endpoints in Z is contained in the k-neighborhood of Z. In particular, up to perhaps increasing the number k, any two points in Z can be connected in Z by a (not necessarily continuous) path which is a k-quasi-geodesic in $\mathcal{G}$.

In Section 5 of [H13] we showed

Theorem 2.3. *Let $\mathcal{G}$ be a metric graph and let $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ be a bounded family of complete connected subgraphs of $\mathcal{G}$. Assume that the following conditions are satisfied.*

1. *There is a number $\delta > 0$ such that each of the graphs H_c is δ-hyperbolic.*
2. *The $\mathcal{H}$-electrification $\mathcal{EG}$ of $\mathcal{G}$ is hyperbolic.*
3. *$\mathcal{H}$ has the bounded penetration property.*

Then $\mathcal{G}$ is hyperbolic. Enlargements of geodesics in $\mathcal{EG}$ are uniform quasi-geodesics in $\mathcal{G}$. The subgraphs H_c are uniformly quasi-convex.

In fact, although this was not stated explicitly, one obtains that the graph $\mathcal{EG}$ is δ'-hyperbolic for a number δ' only depending on the hyperbolicity constant for $\mathcal{EG}$, the common hyperbolicity constant δ for the subgraphs H_c and the constants which enter in the bounded penetration property.

3. Hyperbolicity of the graph of non-separating curves

In this section we consider an arbitrary surface S of genus $g \geq 2$ with $m \geq 0$ punctures. Let $\mathcal{CG}$ be the curve graph of S and let $\mathcal{NC}(S,1)$ be the complete subgraph of $\mathcal{CG}$ whose vertex set consists of non-separating curves. The goal of this section is to show

Proposition 3.1. *The graph $\mathcal{NC}(S,1)$ is hyperbolic.*

Example: If S is a surface of genus $g = 1$ with two punctures then any two non-separating simple closed curves in S intersect and the graph $\mathcal{NC}(S,1)$ does not have edges.

We begin with the following simple lemma.

Lemma 3.2. *If S has at most one puncture then the inclusion $\mathcal{NC}(S,1) \to \mathcal{CG}$ is a 1-quasi-isometry.*

Proof. Let $\gamma : [0,n] \to \mathcal{CG}$ be a simplicial geodesic in $\mathcal{CG}$ connecting two non-separating curves $\gamma(0), \gamma(n)$. We first replace each vertex $\gamma(2i)$ with even parameter which is a separating curve by a non-separating curve as follows.

Assume that $\gamma(2i)$ is separating for some i. Since S has at most one puncture, $\gamma(2i)$ decomposes S into two surfaces S_1, S_2 of positive genus. The curves $\gamma(2i-1), \gamma(2i+1)$ are disjoint from $\gamma(2i)$. If they are contained in distinct components of $S - \gamma(2i)$ then they are disjoint and hence they are connected in $\mathcal{CG}$ by an edge. In particular, we can shorten γ with fixed endpoints. Since γ is length minimizing this is impossible.

Thus $\gamma(2i-1), \gamma(2i+1)$ are contained in the same component of $S - \gamma(2i)$, say in S_1. Replace $\gamma(2i)$ by a non-separating simple closed curve $\tilde{\gamma}(2i)$ contained in S_2. The curve $\tilde{\gamma}(2i)$ is disjoint from both $\gamma(2i-1), \gamma(2i+1)$. Doing this for each i yields a new geodesic $\tilde{\gamma}$ in $\mathcal{CG}$ with the same endpoints so that for all i the curve $\tilde{\gamma}(2i)$ is non-separating. Moreover, we have $\tilde{\gamma}(2i+1) = \gamma(2i+1)$ for all i.

Repeat this construction with the odd parameters. The resulting geodesic is contained in $\mathcal{NC}(S,1)$. $\square$

The case of a surface with at least two punctures is not as straightforward and indeed, we will see below that the vertex inclusion does not define a quasi-isometric embedding of $\mathcal{NC}(S,1)$ into $\mathcal{CG}$.

Assume from now on that $g \geq 2$ and that S has $m \geq 2$ punctures. Define a properly embedded incompressible subsurface X of S to be *thick* if each of the boundary circles of X is separating in S and if moreover there is no non-separating simple closed curve in S which is contained in $S - X$.

If $X \subset S$ is thick then each component of $S - X$ is a bordered punctured sphere with connected boundary. If we collapse each boundary circle of X to a puncture then we can view X as a surface of finite type whose genus equals the genus of S. In particular, we can look at thick subsurfaces of X. However, thick subsurfaces X_0 of X are precisely the thick subsurfaces of S which are contained in X. Moreover, the only thick subsurface of a surface S with at most one puncture is the surface S itself.

For a thick subsurface X of S and for $p \geq 1$ define a graph $\mathcal{A}(X,p)$ as follows. Vertices of $\mathcal{A}(X,p)$ are non-separating simple closed curves in X. Two such vertices c, d are connected by an edge of length one if either they are disjoint or if they are both contained in a proper thick subsurface Y of X of Euler characteristic $\chi(X)+p$. Note that if $p > -\chi(X) - 2g + 2$ then $\mathcal{A}(X,p) = \mathcal{NC}(X,1)$.

Let $\mathcal{CG}(X)$ be the curve graph of X. We have

Lemma 3.3. *For every thick subsurface X of S the vertex inclusion extends to a 2-quasi-isometry $\mathcal{A}(X,1) \to \mathcal{CG}(X)$.*

Proof. Since two simple closed curves which are contained in a proper thick subsurface Y of X are disjoint from a boundary circle of Y which is essential in X, the vertex inclusion $\mathcal{A}(X,1) \to \mathcal{CG}(X)$ is 2-Lipschitz. Thus it suffices to show that the distance in $\mathcal{A}(X,1)$ between any two non-separating simple closed curves does not exceed twice their distance in $\mathcal{CG}(X)$.

To this end let $\gamma : [0,n] \to \mathcal{CG}(X)$ be a simplicial geodesic connecting two non-separating curves $\gamma(0), \gamma(n)$. We proceed as in the proof of Lemma 3.2 and construct first a geodesic $\tilde{\gamma}$ in $\mathcal{CG}(X)$ with the same endpoints such that for each i, the curve $\tilde{\gamma}(i)$ either is non-separating or it decomposes X into a thick subsurface and a three-holed sphere. Call a curve with either of these two properties *admissible* in the sequel.

Replace first each of the vertices $\gamma(2i)$ with even parameter by an admissible curve. Namely, if $\gamma(2i)$ is not admissible then $\gamma(2i)$ decomposes X into two surfaces X_1, X_2 which are different from three holed spheres.

As in the proof of Lemma 3.2 we may assume that $\gamma(2i-1), \gamma(2i+1)$ are contained in say X_1. Then X_2 either has positive genus and hence contains a non-separating curve, or it is a sphere with at least four holes and contains an admissible curve. Thus there is an admissible curve $\tilde{\gamma}(2i) \subset X_2$, and this curve is disjoint from $\gamma(2i-1) \cup \gamma(2i+1)$. Replace $\gamma(2i)$ by $\tilde{\gamma}(2i)$. This process leaves the points $\gamma(2i+1)$ with odd parameter unchanged.

In a second step, replace with the same construction the points $\gamma(2i+1)$ with odd parameter by an admissible curve. Let $\tilde{\gamma} : [0,n] \to \mathcal{CG}$ be the resulting simplicial geodesic. The image of every vertex is admissible.

The geodesic $\tilde{\gamma}$ is now modified as follows. Replace each edge $\tilde{\gamma}[i, i+1]$ connecting two separating admissible simple closed curves by an edge path in $\mathcal{CG}(X)$ of length 2 with the same endpoints so that the middle vertex is a non-separating simple closed curve. This is possible because if c_1, c_2 are two disjoint separating admissible curves then $c_1 \cup c_2$ is disjoint from some non-separating simple closed curve in X. The length of the resulting path $\hat{\gamma}$ is at most twice the length of γ.

The path $\hat{\gamma}$ can be viewed as a path in $\mathcal{A}(X,1)$ by simply erasing all vertices which are separating admissible simple closed curves. Namely, each such vertex is adjacent to two vertices which are non-separating simple closed curves, contained in a proper thick subsurface of S. Thus by the definition of $\mathcal{A}(X,1)$, these curves are connected in $\mathcal{A}(X,1)$ by an edge. This shows that the endpoints of γ are connected in $\mathcal{A}(X,1)$ by a path whose length does not exceed twice the distance in $\mathcal{CG}(X)$ between the endpoints. $\square$

Write $\mathcal{A}(p)=\mathcal{A}(S,p)$. Our goal is to use Lemma 3.3 and induction on p to show that $\mathcal{A}(p)$ is hyperbolic for all p. Since $\mathcal{A}(p)=\mathcal{NC}(S,1)$ for $p>-\chi(S)-2g+2$, this then shows Proposition 3.1. The main technical tool for the induction step is Theorem 2.3.

Let now $p-1\geq 1$ and let X be a thick subsurface of S such that $\chi(X)=\chi(S)+p-1$. Let H_X be the complete subgraph of $\mathcal{A}(p)$ whose vertex set consists of all non-separating simple closed curves which are contained in X. Let $\mathcal{H}=\{H_X\mid X\}$. The next easy observation is the basic setup for the induction step.

Lemma 3.4. *$\mathcal{A}(p-1)$ is 2-quasi-isometric to the $\mathcal{H}$-electrification of $\mathcal{A}(p)$.*

Proof. Let c,d be any two simple closed curves which are connected in $\mathcal{A}(p-1)$ by an edge. The either c,d are disjoint and hence connected in $\mathcal{A}(p)$ by an edge, or c,d are contained in a thick subsurface X of S of Euler characteristic $\chi(X)=\chi(S)+p-1$. This just means that the distance between c,d in the $\mathcal{H}$-electrification $\mathcal{E}$ of $\mathcal{A}(p)$ is at most two. Thus the vertex inclusion $\mathcal{A}(p-1)\to\mathcal{E}$ is two-Lipschitz.

That this is in fact a 2-quasi-isometry follows from the observation that $\mathcal{A}(p)$ is obtained from $\mathcal{A}(p-1)$ by deleting some edges, moreover the endpoints of an embedded simplicial path in $\mathcal{E}$ of length 2 whose midpoint is a special vertex not contained in $\mathcal{A}(p)$ are connected by an edge in $\mathcal{A}(p-1)$. $\square$

Our goal is now to check that the family $\mathcal{H}$ has the properties stated in Theorem 2.3. The following lemma together with Lemma 3.3 implies that the graphs H_X are δ-hyperbolic for a universal constant $\delta>0$.

Lemma 3.5. *H_X is isometric to $\mathcal{A}(X,1)$.*

Proof. If c_1,c_2 are two non-separating simple closed curves in X then c_1,c_2 are connected in $\mathcal{A}(p)$ by an edge if either c_1,c_2 are disjoint or if c_1,c_2 are contained in a thick subsurface X_0 of X of Euler characteristic $\chi(X_0)=\chi(S)+p=\chi(X)+1$. But this is equivalent to stating that c_1,c_2 are connected by an edge in $\mathcal{A}(X,1)$. $\square$

Lemma 3.6. *The family of subgraphs H_X where X runs through the thick subsurfaces of S of Euler characteristic $\chi(S)+p-1$ is bounded.*

Proof. Let X,Y be two thick subsurfaces of S of Euler characteristic $\chi(S)+p-1$. If $X\neq Y$ then up to homotopy, $X\cap Y$ is a (possibly disconnected) subsurface of X whose Euler characteristic is strictly bigger than the Euler characteristic of X. In particular, the diameter in the curve graph of X of the set of simple closed curves contained in $X\cap Y$ is uniformly bounded. Thus the lemma follows from Lemma 3.3 and Lemma 3.5. $\square$

The proof of the bounded penetration property is more involved. To this end recall from [MM00] that for every proper connected subsurface X of S there is a *subsurface projection* π_X of $\mathcal{CG}$ into the subsets of the arc and curve graph of

X. This projection associates to a simple closed curve c in S the intersection components $\pi_X(c)$ of c with X, viewed as a subset of the arc and curve graph of X. The diameter of the image is at most one. If c is disjoint from X then this projection is empty. The arc and curve graph of X is 2-quasi-isometric to the curve graph of X (see [MM00]).

We need the following result from [MM00] (in the version formulated in Lemma 6.5 of [H13]).

Proposition 3.7. *For every number $L > 1$ there is a number $\xi(L) > 0$ with the following property. Let Y be a proper connected subsurface of S and let γ be a simplicial path in $\mathcal{CG}$ which is an L-quasi-geodesic. If $\pi_Y(v) \neq \emptyset$ for every vertex v on γ then*

$$\mathrm{diam}\pi_Y(\gamma) < \xi(L).$$

If $\gamma : [0, n] \to \mathcal{A}(S, 1)$ is any geodesic then for all j, the curves $\gamma(j)$ and $\gamma(j+1)$ either are disjoint and hence connected in $\mathcal{CG}$ by an edge, or they are contained in a common thick subsurface Y of S of Euler characteristic $\chi(S) + 1$. In the second case replace the edge $\gamma[j, j+1]$ by an edge path in $\mathcal{CG}$ of length two connecting the same endpoints which passes through an essential simple closed curve in the complement of Y. We call $\tilde{\gamma}$ a *canonical modification* of γ. By Lemma 3.3 and its proof, $\tilde{\gamma}$ is a simplicial path in $\mathcal{CG}$ which is a 2-quasi-geodesic.

We next define a family of geodesics in $\mathcal{A}(S, 1)$ which serve as substitutes for the tight geodesics as introduced in [MM00]. Namely, for numbers $\kappa > 0, p \geq 1$ define a simplicial path $\zeta : [0, n] \to \mathcal{A}(S, 1)$ to be *(κ,p)-good* if the following holds true. Let $X \subset S$ be any thick subsurface of Euler characteristic $\chi(X) \geq \chi(S) + p$; then there is a number $u = u(X) \in (0, n)$ with the following property.

(1) For every $j \leq u$, $\mathrm{diam}(\pi_X(\zeta(0) \cup \zeta(j))) \leq \kappa$.
(2) For every $j > u$, $\mathrm{diam}(\pi_X(\zeta(j) \cup \zeta(n))) \leq \kappa$.

Thus in a good path, big subsurface projections can be explicitly localized.

We use Proposition 3.7 to show

Lemma 3.8. *There is a number $\kappa_1 > 0$ such that any two vertices in $\mathcal{A}(S, 1)$ can be connected by a $(\kappa_1, 1)$-good geodesic.*

Proof. Let c_1, c_2 be non-separating simple closed curves and let $\gamma : [0, n] \to \mathcal{A}(S, 1)$ be a geodesic connecting c_1 to c_2, with canonical modification $\tilde{\gamma} : [0, \tilde{n}] \to \mathcal{CG}$.

Let $\ell_1 < \cdots < \ell_s$ be such that for each i the curves $\tilde{\gamma}(\ell_i), \tilde{\gamma}(\ell_i + 2)$ are both separating and such that the subsurface of S filled by $\tilde{\gamma}(\ell_i) \cup \tilde{\gamma}(\ell_i + 2)$ is a holed sphere whose complement Z is thick (this set may be empty). Let $\tilde{\gamma}_1(\ell_i + 1)$ be a non-separating simple closed curve contained in Z which is of distance one to the subsurface projection $\pi_Z(c_2)$ of c_2.

The simplicial path $\tilde{\gamma}_1$ constructed in this way is a canoncial modification of a geodesic γ_1 in $\mathcal{A}(S, 1)$ connecting c_1 to c_2. We claim that γ_1 is a $(\xi(2), 1)$-good geodesic in $\mathcal{A}(S, 1)$ where $\xi(2) > 0$ is as in Proposition 3.7.

Namely, if Z is an arbitrary thick subsurface of S then since γ_1 is a geodesic, there are at most two parameters $k, k + \iota$ (here $\iota = 0$ or $\iota = 2$) such that $\tilde{\gamma}_1(k), \tilde{\gamma}_1(k + \iota)$ is disjoint from Z. Since $\tilde{\gamma}_1$ is a 2-quasi-geodesic in $\mathcal{CG}$, if there is at most one such point (which is in particular the case if the Euler characteristic of Z equals $\chi(S) + 1$) then the properties (1),(2) for $\xi(2)$ are immediate from Lemma 3.3 and Proposition 3.7. Otherwise the property follows from the construction of γ_1 and

the fact that for subsurfaces $X \subset Y \subset S$ and any simple closed curve c we have $\pi_X(c) = \pi_X(\pi_Y(c))$ (with a small abuse of notation). $\square$

We use Proposition 3.7 and Lemma 3.8 to define a *level p hierarchy path* in $\mathcal{A}(p)$ connecting two non-separating simple closed curves c_1, c_2 as follows. The starting point is a $(\kappa_1, 1)$-good geodesic $\gamma : [0, n] \to \mathcal{A}(S, 1)$. For any j so that the curves $\gamma(j), \gamma(j+1)$ are not disjoint there is a thick subsurface Y_j of Euler characteristic $\chi(Y_j) = \chi(S) + 1$ so that $\gamma(j), \gamma(j+1) \subset Y_j$. Replace the edge $\gamma[j, j+1]$ by a $(\kappa_1, 1)$-good geodesic in $\mathcal{A}(Y_j, 1)$. The resulting path is contained in the subgraph $\mathcal{A}(2)$ of $\mathcal{A}(1)$. Proceed inductively and construct in p such steps a simplicial path in $\mathcal{A}(p) \subset \mathcal{A}(1)$ connecting c_1 to c_2 which we call a *level p hierarchy path*.

Lemma 3.9. *For every $p \geq 1$ there is a number $\kappa_p > 0$ such that a level p hierarchy path in $\mathcal{A}(p)$ is (κ_p, p)-good.*

Proof. We proceed by induction on p. The case $p = 1$ follows from the definition of a hierarchy path and Lemma 3.8. Thus assume that the lemma holds true for all $p - 1 \geq 1$.

Let $\gamma : [0, n] \to \mathcal{A}(p)$ be a level p hierarchy path. By construction, γ is obtained from a level $p-1$ hierarchy path $\zeta : [0, s] \to \mathcal{A}(p-1)$ as follows. There are numbers $0 < \tau_1 < \cdots < \tau_q < s$ such that for each i, the edge $\zeta[\tau_i, \tau_i + 1]$ connects two non-separating simple closed curves which are contained in a thick subsurface Z_i of S of Euler characteristic $\chi(S) + p - 1$. The arc γ is obtained from ζ by replacing each of the edges $\zeta[\tau_i, \tau_i + 1]$ by a $(\kappa_1, 1)$-good geodesic in $\mathcal{A}(Z_i, 1)$.

By induction hypothesis, ζ is $(\kappa_{p-1}, p-1)$-good for a number $\kappa_{p-1} > 1$ not depending on ζ. Thus for any thick subsurface Z of S of Euler characteristic $\chi(Z) \geq \chi(S) + p$ there is a number $u \in [0, s]$ so that for all $j \leq u$ we have $\mathrm{diam}(\pi_Z(\zeta(0), \zeta(j))) \leq \kappa_{p-1}$ and similarly for $j \geq u + 1$.

Let now $i > 0$ be such that $\tau_i < u$. There is a subarc ρ of γ which is a $(\kappa_1, 1)$-good geodesic connecting $\zeta(\tau_i)$ to $\zeta(\tau_i + 1)$ in Z_i. Since by construction ρ is a $(\kappa_1, 1)$-good geodesic in $\mathcal{A}(Z_i, 1)$ and since $\mathrm{diam}(\pi_Z(\zeta(\tau_i) \cup \zeta(\tau_i + 1))) \leq 2\kappa_{p-1}$, for each vertex $\rho(t)$ the diameter of the subsurface projection into Z of $\zeta(\tau_i) \cup \rho(t)$ does not exceed $2\kappa_1$. Then for each t we have

$$\mathrm{diam}(\pi_Z(\zeta(0) \cup \rho(t))) \leq \kappa_{p-1} + 2\kappa_1 = \kappa_p.$$

This argument is also valid for $\tau_i > 0$.

Finally if $\tau_i = u$ then we can apply the same reasoning as before to the κ_1-good geodesic ρ and obtain the statement of the lemma. $\square$

Proof of Proposition 3.1: By Lemma 3.2, if S has at most one puncture then the inclusion $\mathcal{NC}(S, 1) \to \mathcal{CG}$ is a quasi-isometry.

If the number of punctures is at least two then we show by induction on p the following.

a) The graph $\mathcal{A}(p)$ is hyperbolic.
b) Level p hierarchy paths are uniform quasi-geodesics in $\mathcal{A}(p)$.
c) For every $L > 1$ there is a number $\xi(L, p) > 0$ with the following property. Let Y be a thick subsurface of S of Euler characteristic $\chi(Y) \geq \chi(S) + p$ and let γ be a simplicial path in $\mathcal{A}(p)$ which is an L-quasi-geodesic. Let $\tilde{\gamma}$ be the canonical modification of γ. If $\pi_Y(v) \neq \emptyset$ for every vertex v on $\tilde{\gamma}$ then $\mathrm{diam}\pi_Y(\gamma) < \xi(L, p)$.

The case $p=1$ follows from Lemma 3.3 and Proposition 3.7 Assume that the claim holds true for $p-1\geq 1$.

For a thick subsurface X of Euler characteristic $\chi(X)=\chi(S)+p-1$ let as before H_X be the complete subgraph of $\mathcal{A}(p)$ whose vertex set consists of all non-separating curves contained in X and let $\mathcal{H}=\{H_X\mid X\}$. By Lemma 3.4, $\mathcal{A}(p-1)$ is 2-quasi-isometric to the $\mathcal{H}$-electrification of $\mathcal{A}(p)$. Moreover by construction, level p hierarchy paths are enlargements of level $p-1$ hierarchy paths. Therefore by the induction hypothesis, to establish properties a),b) above for p it suffices to show that the family $\mathcal{H}$ is bounded, satisfies the assumptions (1),(3) in the statement of Theorem 2.3.

Lemma 3.6 shows that the family $\mathcal{H}=\{H_X\mid X\}$ is bounded.

By Lemma 3.5, H_X is isometric to $\mathcal{A}(X,1)$ and hence by Lemma 3.3, $\mathcal{A}(X,1)$ is δ-hyperbolic for a number $\delta>0$ not depending on X. The bounded penetration property for $\mathcal{H}$ follows from property c) above, applied to thick subsurfaces of Euler characteristic $\chi(S)+p$. Thus by Theorem 2.3 and the induction hypothesis, $\mathcal{A}(p)$ is hyperbolic and level p hierarchy paths are uniform quasi-geodesics in $\mathcal{A}(p)$.

We are left with verifying property c) above. By Lemma 3.9, this property holds true for level p hierarchy paths with the number $\kappa_p>0$ replacing $\xi(L,p)$. The argument in the proof of Lemma 6.5 of [H13] then yields this property for an arbitrary efficient L-quasi-geodesic in $\mathcal{A}(p)$ for a suitable number $\xi(L,p)>0$.

Namely, by hyperbolicity, for every $L>1$ there is a number $n(L)>1$ so that for every L-quasi-geodesic $\eta:[0,n]\to\mathcal{A}(p)$ of finite length, the Hausdorff distance between the image of η and the image of a level p hierarchy path γ with the same endpoints does not exceed $n(L)$.

Let $Y\subset S$ be a thick subsurface of Euler characteristic $\chi(Y)\geq\chi(S)+p$. Assume that

$$\mathrm{diam}(\pi_Y(\eta(0)\cup\eta(n)))\geq 2\kappa_p+L(2n(L)+4).$$

By the properties of level p hierarchy paths, if $\tilde\gamma$ denotes the canonical modification of γ then there is some $u\in\mathbb{Z}$ so that $\tilde\gamma(u)\in A$ where $A\subset\mathcal{CG}$ is the set of all curves which are disjoint from Y. By construction, the canonical modification $\tilde\eta$ of η passes through the $n(L)$-neighborhood of A. By this we mean that there is a point x on $\tilde\eta$ and a canonical modification of simplicial path in $\mathcal{A}(p)$ of length at most $n(L)$ which connects x to a point in A.

Let $s+1\leq t-1$ be the smallest and biggest number, respectively, so that $\tilde\eta(s+1),\tilde\eta(t-1)$ are contained in the $n(L)$-neighborhood of A. A canonical modification of a level p hierarchy path connecting $\tilde\eta(0)$ to $\tilde\eta(s)$ does not pass through A and similarly for a canonical modification of a level p hierarchy path connecting $\tilde\eta(t)$ to $\tilde\eta(n)$. This implies that

$$\mathrm{diam}(\pi_Y(\tilde\eta(s)\cup\tilde\eta(t)))\geq L(2n(L)+4).$$

Now the distance in $\mathcal{A}(p)$ between $\gamma(s),\gamma(t)$ is at most $2n(L)$, and if c,d are disjoint simple closed curves which intersect Y then the diameter of $\pi_Y(c\cup d)$ is at most one. Thus if $\tilde\eta(\ell)$ intersects Y for all ℓ then the diameter of the subsurface projection of the endpoints is at most $L(2n(L)+3)$ which is a contradiction.

This completes the induction step and proves the proposition. $\square$

The arguments in [H13] can now be used without modification to identify the Gromov boundary of $\mathcal{NC}(S,1)$. To this end let $\mathcal{L}$ be the set of all geodesic laminations on S equipped with the coarse Hausdorff topology. In this topology, a

sequence (ν_i) converges to ν if any limit in the usual Hausdorff topology of a convergent subsequence contains ν as a sublamination.

For each thick subsurface X of S let $\mathcal{L}(X) \subset \mathcal{L}$ be the set of all minimal geodesic laminations which fill up X, equipped with the coarse Hausdorff topology. We have

Corollary 3.10. *The Gromov boundary of $\mathcal{NC}(S,1)$ equals $\cup_X \mathcal{L}(X)$ equipped with the coarse Hausdorff topology.*

4. Proof of the theorem

The proof of the Theorem from the introduction follows as in Section 3 from Proposition 3.1 and an induction. There are no new tools needed, and we will only sketch those parts of the argument which were discussed in detail in Section 3.

Proposition 4.1. *For $n < g$ the graph $\mathcal{NC}(S,n)$ is hyperbolic.*

Proof. We proceed by induction on n. The case $n = 1$ was shown in Proposition 3.1. Thus assume that the proposition is known whenever $n - 1 \geq 1$. Consider the graph $\mathcal{NC}(S,n)$.

For an $n-1$-tuple ν of non-separating simple closed curves so that $S - \nu$ is connected let H_ν be the complete subgraph of $\mathcal{NC}(S,n)$ whose vertex set consists of all non-separating n-multi-curves containing ν. Clearly H_ν can naturally be identified with the graph of all nonseparating simple closed curves in $S - \nu$. Since the genus of $S - \nu$ is at least two, by Proposition 3.1 the graph is δ-hyperbolic for a number $\delta > 0$ not depending on ν. Let $\mathcal{H} = \{H_\nu \mid \nu\}$.

We claim that $\mathcal{NC}(S,n-1)$ is quasi-isometric to the $\mathcal{H}$-electrification $\mathcal{E}$ of $\mathcal{NC}(S,n)$. Namely, define a vertex embedding $\Lambda : \mathcal{NC}(S,n-1) \to \mathcal{E}$ by associating to a non-separating $(n-1)$-multicurve c any non-separating n-multicurve $\Lambda(c)$ containing c. We claim that Λ is coarsely 6-Lipschitz.

To see this let c_0, c_1 be connected by an edge in $\mathcal{NC}(S,n-1)$. Then c_1 is obtained from c_0 by removing a component a from c_0 and replacing it by a component b disjoint from c_0.

The union $c_0 \cup b$ is a multi-curve with n components. If this multi-curve is non-separating then we can view it as a vertex $x \in \mathcal{E}$. Since $\Lambda(c_0)$ contains c_0, the distance in $\mathcal{E}$ between $\Lambda(c_0)$ and x equals at most 2. Similarly, the distance in $\mathcal{E}$ between $\Lambda(c_1)$ and x is at most 2 and hence the distance in $\mathcal{E}$ between $\Lambda(c_0)$ and $\Lambda(c_1)$ is at most 4.

If $c_0 \cup b$ is not non-separating then $a \cup b$ is a bounding pair in $S - (c_0 - a)$. Choose a non-separating simple closed curve ω disjoint from a, b, c_0 so that both $c_0 \cup \omega$ and $c_1 \cup \omega$ are non-separating and apply the same argument to the non-separating $(n-1)$-multicurves $c_0, (c_0 - a) \cup \omega$ and to $(c_0 - a) \cup \omega, c_1$. We conclude that the distance in $\mathcal{E}$ between $\Lambda(c_0)$ and $\Lambda(c_1)$ is at most 6.

On the other hand, a map which associates to a vertex $x \in \mathcal{E}$ an $(n-1)$-multicurve contained in x is a coarsely Lipschitz coarse inverse of Λ. Thus indeed Λ is a quasi-isometry.

By the above discussion and the induction hypothesis, we can complete the induction step with an application of Theorem 2.3. To this end we have to check that the assumptions in the theorem are satisfied.

For $\nu \neq \zeta \in \mathcal{NC}(S,n-1)$ consider the intersection $H_\nu \cap H_\zeta$; this intersection consists of all non-separating n-multicurves which contain both ν and ζ and hence this intersection consists of at most one point. Thus $\mathcal{H}$ is bounded.

We are left with the bounded penetration property. This is done exactly as in the proof of Proposition 3.1, using a notion of good simplicial paths which control subsurface projections into subsurfaces of S which are obtained from some thick subsurface by removing a non-separating multicurve. Hierarchy paths as defined in Section 3 are good by construction, and these paths can successively be enlarged as discussed in Section 3 to good paths in this sense.

Thus all requirements in Theorem 2.3 are satisfied. In particular, the graph $\mathcal{NC}(S,n)$ is hyperbolic if its $\mathcal{H}$-electrification is.

But the $\mathcal{H}$-electrification is quasi-isometric to the graph $\mathcal{NC}(S,n-1)$ and hence it is hyperbolic by induction hypothesis. The proposition follows. □

Define a subsurface Y of S to be *p-semi-thick* if Y is the complement in a thick subsurface X of S of a non-separating multicurve with at most $p-1$ components. Let $\mathcal{L}(Y)$ be the set of all minimal geodesic laminations which fill up Y. Similarly to Corollary 3.10 we have

Corollary 4.2. *The Gromov boundary of $\mathcal{NC}(S,p)$ equals $\cup_Y \mathcal{L}(Y)$ equipped with the coarse Hausdorff topology.*

Remark: The main result in this note can also be obtained with the tools developed in [MS13]. To the best of our knowledge, these tools do not have any obvious advantage over the tools we used.

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